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Խաստյան Էրիկ Արթուրի

Կելերյան փուլային տարածություններով սուպերսիմետրիկ մեխանիկաներ

Ա.04.02 - «Տեսական ֆիզիկա» մասնագիտությամբ ֆիզիկա-մաթեմատիկական գիտությունների թեկնածուի գիտական աստիճանի հայցման ատենախոսության

ՍԵՂՄԱԳԻՐ

ԵՐԵՎԱՆ - 2024

A. I. ALIKHANYAN NATIONAL SCIENCE LABORATORY
(YEREVAN PHYSICS INSTITUTE)

Erik Khastyan

Supersymmetric mechanics with Kähler phase space

SYNOPSIS

of Dissertation in 01.04.02 - Theoretical physics presented for the degree of candidate in physical and mathematical sciences

YEREVAN - 2024

Ատենախոսության թեման հաստատվել է Ա. Ի. Ալիխանյանի անվան Ազգային Գիտական Լաբորատորիայի (ԵրՖԻ) գիտական խորհուրդում:

Գիտական ղեկավար՝	
Ֆիզ. մաթ. գիտ. դոկտոր	Ներսեսյան Արմեն Պետրոսի (ԱԱԳԼ)
Պաշտոնական ընդդիմախոսներ՝	
Ֆիզ. մաթ. գիտ. դոկտոր	Պողոսյան Ռուբիկ Հրաչիկի (ԱԱԳԼ)
Ֆիզ. մաթ. գիտ. դոկտոր	Հակոբյան Տիգրան Ստեփանի (ԵՊՀ)
Առաջատար կազմակերպություն՝	Համբուրգի համալսարան, Տեսական ֆիզիկայի II ինստիտուտ, (Գերմանիա)

Ատենախոսության պաշտպանությունը կայանալու է 2024 թ. Ապրիլի 19 -ին, ժամը 14:00-ին, ԱԱԳԼ-ում գործող ԲԿԳԿ-ի 024 «Ֆիզիկայի» մասնագիտական խորհուրդում (Երևան, 0036, Ալիխանյան Եղբայրների փ. 2):

Ատենախոսությանը կարելի է ծանոթանալ ԱԱԳԼ-ի գրադարանում:
Սեղմագիրն առաքված է 2024 թ. մարտի 19-ին:

Մասնագիտական խորհրդի գիտական քարտուղար՝
Ֆիզ. մաթ. գիտ. դոկտոր — Հրաչյա Մարուքյան

The subject of the dissertation is approved by the scientific council of the A.I. Alikhanyan National Science Laboratory (YerPhI).

Scientific supervisor:	
Doctor of ph-math. sciences	Armen Nersessian (AANL)
Official opponents:	
Doctor of ph-math. sciences	Rubik Poghossian (AANL)
Doctor of ph-math. sciences	Tigran Hakobyan (YSU)
Leading organization:	Hamburg University, II. Institute for Theoretical Physics (Germany)

The defense will take place on the 19 of April 2024 at 14:00 during the ”Physics” professional council’s session of HESC 024 acting within AANL (2 Alikhanyan Brothers str., 0036, Yerevan).

The dissertation is available at the AANL library.
The synopsis is sent out on the 19 of March, 2024.

Scientific secretary of the special council:
Doctor of ph-math. sciences — Hrachya Marukyan

Abstract

The dissertation is devoted to the study of Kähler manifolds as the phase space of classical mechanical (super)integrable systems. Kähler manifolds are widely used in numerous sections of modern theoretical and mathematical physics. They are mostly considered as the configuration space of various systems. Considering them as the phase space of some (super)integrable systems leads us to an interesting and elegant description of integrability rooted in the underlying geometry of the phase space.

As demonstrated in the dissertation, the methods used herein can be extremely useful for the supersymmetrization of a given model. The phase superspaces of such supersymmetric extensions are Kähler super-manifolds, meaning a Kähler manifold equipped with Grassmann anticommuting coordinates as well.

The thesis comprises four chapters (excluding the introductory chapter). The second chapter is dedicated to a renowned classical system known as the Euler top. It is considered as a one-dimensional system with a Kähler phase space, specifically, $\mathbb{CP}^1$, and its super-symmetrization in that context. In the third chapter, we study compact and non-compact complex projective spaces, along with their symmetries. Focusing on the non-compact case, we regard them as phase spaces of N-dimensional conformal mechanics, N-dimensional oscillator, and N-dimensional Coulomb system. Their integrability properties are studied from the geometrical point of view. In the fourth chapter, we examine the possibilities of supersymmetrization within this formalism and consider the options for supersymmetrization for the examples from the previous chapter. Finally, the fifth chapter is devoted to the discussion of the main results and possible future developments of the ideas explored in this thesis.

Relevance and Motivation

One of the advantages of considering integrable systems on Kähler manifolds is that the Kähler structure of the phase spaces enables the use of the geometric quantization method. The number of known nontrivial (super)integrable systems featuring a Kähler phase space is quite limited, and their examination remains at the periphery of integrable systems theory. This is particularly surprising, given that the quantization of systems with a Kähler phase space has been a focal point in modern geometry ever since the inception of geometric quantization. A notable integrable model with a Kähler phase space that is currently under extensive investigation is the (compactified) Ruijesnaars-Schneider model. However, even this system is primarily studied in canonical coordinates.

Establishing a connection between existing integrable systems and their constants of motion with the isometries of a Kähler manifold viewed as a phase space can be useful in comprehending the system's geometry. It's an important step towards quantization in non-canonical coordinates. Besides, there are indications that Kähler phase spaces can be useful for studying conventional Hamiltonian systems, particularly those formulated on the cotangent bundle of Riemann manifolds.

Aim of dissertation

Developing a geometrical formalism for studying (super)integrable systems. Analysis of the possibility of construction of supersymmetric extensions of a given (super)integrable model within that formalism.

Publications and Conferences

The dissertation is primarily based on four papers published in Physical Review D, Physics Letters A, International Journal of Modern Physics A, and Physics of Particles and Nuclei Letters. Further details can be found at the end of this synopsis in the publication list.

Throughout the project, six talks were delivered on the topic at various conferences:

- Supersymmetry in Integrable Systems, Dubna (2023)
- Recent Advances in Fundamental Physics, Tbilisi (2022)
- Symmetry Methods in Physics, Yerevan (2022)
- Supersymmetry and Integrability, Dubna (2022)
- Aspects of Symmetry, Online (2021)
- Recent Advances in Mathematical Physics, Online (2020)

Main points to defend

- The Euler top was formulated as a one-dimensional system with a phase space $\mathbb{CP}^1$. Then we proposed the procedure of $\mathcal{N}=2k$ *a priori* integrable supersymmetrization of a generic one-dimensional systems which provides the family of $\mathcal{N}$-supersymmetric extensions depending on $\mathcal{N}/2$ arbitrary real functions. Thus, we gave the $\mathcal{N}=2k$ supersymmetric extensions of the Euler top as well.
- The superintegrable generalizations of conformal mechanics, oscillator and Coulomb systems can be naturally described in terms of the non-compact complex projective space considered as a phase space.
- The $su(1,N|M)$-superconformal mechanics was constructed, formulating them on phase superspace given by the non-compact analog of complex projective superspace $\mathbb{CP}^{N|M}$.
- Superintegrable oscillator- and Coulomb- like systems with a $su(1,N|M)$ dynamical superalgebra was proposed. It was found that oscillator-like systems admit deformed $\mathcal{N}=2M$ Poincarè supersymmetry, in contrast with Coulomb-like ones.

Structure of dissertation

The dissertation consists five chapters:

1. Introduction
2. Euler top and freedom in supersymmetrization of one-dimensional mechanics
3. Non-compact complex projective space as a phase space of superintegrable systems
4. $su(1.N|M)$-Superconformal Mechanics and Deformations
5. Discussion

Chapter 1

In this introductory chapter, we study Kähler manifolds in general and also consider some examples that appear in the dissertation. Additionally, some notations are introduced that are common throughout the thesis.

According to Darboux's theorem, any symplectic structure can locally be presented in the canonical form corresponding to canonical Poisson brackets. Furthermore, any cotangent bundle of a Riemann manifold can be equipped with the globally defined canonical symplectic structure. Hence, for the Hamiltonian description of systems of particles moving on the Riemann space, we can restrict ourselves to the canonical symplectic structure (and canonical Poisson brackets). Non-canonical Poisson brackets are usually used for the description of more sophisticated systems, such as various modifications of tops, (iso)spin dynamics, etc.

As it was mentioned above, Kähler manifolds have three mutually compatible structures, namely complex structure, Riemannian structure and symplectic structure. Kähler manifold is a particular case of the general Hermitian manifold ($g_{a\bar{b}}dz^a d\bar{z}^b$). For any Hermitian metric one can define a 2-form

$$\omega = \imath g_{a\bar{b}} dz^a \wedge d\bar{z}^b$$

This 2-form is called a fundamental form.

Hermitian manifold is called if this 2-form is symplectic (closed and non-degenerate). This condition puts a strong limitation, which allows us to express a Kähler metric as the second derivative of a function known as the Kähler potential.

$$g_{a\bar{b}} = \frac{\partial^2 K(z,\bar{z})}{\partial z^a \partial \bar{z}^b}$$

It is worth to mention that it is defined up to a holomorphic or antiholomorphic function: $K(z,\bar{z}) \to K(z,\bar{z}) + U(z) + \bar{U}(\bar{z})$.

Symplectic structure of Kähler manifolds allows as naturally equip it with Poisson brackets.

$$\{f,g\}_0 = i g^{a\bar{b}}\Big(\frac{\partial f}{\partial z^a}\frac{\partial g}{\partial \bar{z}^b} - \frac{\partial g}{\partial z^a}\frac{\partial f}{\partial \bar{z}^b}\Big), \quad g^{a\bar{b}} g_{\bar{b}c} = \delta^a_c\,.$$

Since the symplectic structure relates functions (Hamiltonians) and vector fields (Hamiltonian vector fields), we can introduce functions, which generate Killing vector fields.

$$\mathbf{V}_\mu = \{h_\mu, \}_0 = V^a_\mu \frac{\partial}{\partial z^a} + \bar{V}^{\bar{a}}_\mu \frac{\partial}{\partial \bar{z}^a}, \qquad V^a_\mu = -\imath g^{a\bar{b}} \partial_{\bar{b}} h_\mu$$

Such functions are called Killing potentials. By employing the Killing Equations, limitations on Killing potentials can be deduced. These potentials must have real values and must satisfy the following equation.

$$\frac{\partial^2 h_\mu}{\partial z^a \partial z^b} - \Gamma^c_{ab} \frac{\partial h_\mu}{\partial z^c} = 0$$

These functions are extremely useful for studying systems on Kähler manifolds in presence of a constant magnetic field. Since any 2-form is closed in two (real) dimensions, a one-dimensional orientable complex manifold (Riemann surface) can always be equipped with a Kähler structure.

Many components of the Christoffel symbols and Riemann tensor vanish.

$$\Gamma^a_{bc} = g^{a\bar{d}} g_{b\bar{d},c}, \qquad R^a_{bc\bar{d}} = -(\Gamma^a_{bc})_{,\bar{d}}\,.$$

Chapter 2

In this chapter, we explore a basic one-dimensional model that possesses a Kähler phase space. The model under consideration is an Euler top, a classical mechanical integrable system widely studied for its interesting dynamics. The Euler top represents a rotating rigid body, exhibiting rich behaviors related to angular momentum and precession.

Conventionally it is described by the Hamiltonian system with degenerated Poisson brackets parameterized by the components of angular momentum $\ell = (x_1, x_2, x_3)$,

$$\{x_i, x_j\} = \varepsilon_{ijk} x_k, \qquad H = \sum_{i=1}^{3} \frac{x_i^2}{2I_i},$$

where \$H\$ is the Hamiltonian, and \$I_i>0\$ are the principal momenta of inertia.

Since x_i form $so(3)$ algebra, the system has a Casimir function

$$C = \sum_{i=1}^{3} x_i^2 \ : \quad \{C(x), x_i\} = 0.$$

Its fixation leads to the Hamiltonian system with two-dimensional **non-degenerated** phase space, i.e. one-dimensional system. Hence, Euler top is *a priori* integrable.

For the description of the Euler top in terms of non-degenerated phase space, let us introduce instead of x_i, the coordinates $j, z, \bar{z}$.

$$j := \sqrt{\sum_{i=1}^{3} x_i^2}, \qquad z := \frac{x_1 + \imath x_2}{j - x_3}$$

Clearly, j is the complete angular momentum. In these coordinates the Poisson brackets read

$$\{\bar{z}, z\} = -\frac{\imath}{2j}(1 + z\bar{z})^2, \quad \{z, j\} = \{z, z\} = 0,$$

while the momentum generators look as follows

$$
j^2 := \sum_{i=1}^{3} x_i^2,
$$

$$
x_1 := h_1 = j\,\frac{z+\bar z}{1+z\bar z}, \quad x_2 := h_2 = j\,\frac{\imath(\bar z - z)}{1+z\bar z}\,, \quad x_3 := h_3 = j\,\frac{z\bar z - 1}{1+z\bar z}\,.
$$

Fixing j to be constant we arrive at the two-dimensional phase space (parameterized by the single complex coordinate z and equipped with the one-(complex)dimensional Kähler structure, i.e. the complex projective plane $\mathbb{CP}^1$ with the Fubini-Study metrics and corresponding Kähler potential

$$
g(z,\bar z)dz d\bar z := 2j\,\frac{dz d\bar z}{(1+z\bar z)^2}, \qquad K(z,\bar z) = 2j\,\log(1+z\bar z)\,.
$$

In these terms the Hamiltonian of Euler top reads

$$
H = -j^2\frac{b(z^2+\bar z^2)+2az\bar z}{2(1+z\bar z)^2} + \frac{j^2}{2I_3}, \quad \text{with} \quad a := \frac{2}{I_3} - \frac{1}{I_1} - \frac{1}{I_2}, b := \frac{1}{I_2} - \frac{1}{I_1}\,.
$$

We can rewrite Euler top in canonical coordinates

$$
z = \cot\frac{\theta}{2}\mathrm{e}^{\imath\varphi}, \quad \{\varphi, j\cos\theta\} = 1 \Rightarrow \qquad p := j\cos\theta\,.
$$

Performing a canonical transformation $(p,\varphi)\to(P,Q)$

$$
P = \sqrt{\frac{a+b\cos 2\varphi}{2}}\;p,
$$

$$
Q = \sqrt{\frac{2}{a+b}}\int\frac{d\varphi}{\sqrt{1-\frac{2b}{a+b}\sin^2\varphi}} = \sqrt{\frac{2}{a+b}}F\left(\varphi,k\right): \quad \{Q,P\} = 1,
$$

where $F\left(\varphi,k\right)$ is an elliptic integral of the first kind, with $k = \sqrt{2b/(a+b)}$ being its modulus, and φ is the so-called Jacobi amplitude

$$
\varphi = F^{-1}\left(F,k\right) = \mathrm{amp}\left(F,k\right), \qquad \sin\varphi = \sin(\mathrm{amp}(F,k)) = \mathbf{sn}(F,k),
$$

we arrive to the following form of Hamiltonian

$$
H = \frac{1}{2}P^2 + \frac{j^2 b}{2}\mathrm{sn}^2\left(\sqrt{\frac{a+b}{2}}Q, \sqrt{\frac{2b}{a+b}}\right) + \frac{j^2}{2I_1}\,.
$$

So, the Euler top is the one-dimensional Hamiltonian system with $\mathbb{CP}^1$ phase space and with the Hamiltonian given by the quadratic functions of its Killing potentials. In the canonical coordinates it results in the one-dimensional nonlinear oscillator.

We are interested in the supersymmetrization compatible with the Kähler geometry describing the phase space of the Euler top. There are many ways of supersymmetrization of such a one dimensional systems including Euler top, but here we follow a less geometrical approach. We will consider the systems with generic two-(real)dimensional phase space. Such phase spaces can be always equipped with the one-(complex) dimensional Kähler structure, so that the Poisson brackets will be given by the relation

$$\{z, \bar{z}\} = \frac{\imath}{g(z, \bar{z})}.$$

For the construction of $\mathcal{N}$-supersymmetric extensions of these systems (with even $\mathcal{N}$) we extend this phase space by the canonical complex Grassmann variables ψ_a, $a = 1, \ldots, \frac{\mathcal{N}}{2}$

$$\left\{\psi_a, \bar{\psi}^b\right\} = \imath\,\delta_a^b.$$

With these Poisson brackets at hands we can construct the $\mathcal{N}$ supersymmetric extensions of two-dimensional systems defined by the Poisson brackets $\{z, \bar{z}\}$ and by any positive Hamiltonian $H(z, \bar{z}) > 0$,

$$\left\{Q_a, \bar{Q}^b\right\} = \imath\,\delta_a^b \mathscr{H}, \qquad \mathscr{H} := H(z, \bar{z}) + \text{ fermions}.$$

In accordance with the generalization of Liouville theorem to the supermanifolds these supersymmetric extensions will be *a priory* integrable.

For the construction of $\mathcal{N} = 2$ supersymmetric extension of the system with Hamiltonian $H(z, \bar{z}) > 0$ we choose an appropriate Ansatz for supercharges and arrive the family of $\mathcal{N} = 2$ supersymmetric extensions of the Hamiltonian H, parameterized by the arbitrary real function $\Phi(z, \bar{z})$

$$Q = \sqrt{H}\mathrm{e}^{\imath\Phi}\psi, \qquad \bar{Q} = \sqrt{H}\mathrm{e}^{-\imath\Phi}\bar{\psi} \quad \Rightarrow \quad \mathscr{H} = H + \{\Phi, H\}\psi\bar{\psi}.$$

Specifying the Poisson brackets and Hamiltonian we will get the respective supersymmetric extension of the Euler top.

For the construction of nontrivial $\mathcal{N} = 4$ supersymmetric system we choose the following Ansatz for supercharges

$$Q_a = f_1(z, \bar{z})\psi_a + f_2(z, \bar{z})\psi_a \sum_{b=1}^{\mathcal{N}/2} \psi_b \bar{\psi}^b, \quad \bar{Q}^a = \bar{f}_1(z, \bar{z})\bar{\psi}^a - \bar{f}_2(z, \bar{z})\bar{\psi}^a \sum_{b=1}^{\mathcal{N}/2} \bar{\psi}^b \psi_b, \text{ with}$$

$$f_1(z, \bar{z}) := \sqrt{H}\mathrm{e}^{\imath\Phi_1(z,\bar{z})}, \quad f_2 = R(z, \bar{z})\mathrm{e}^{\imath(\Phi_1 - \Phi_2)},$$

Then, we require that the supercharges form the $\mathcal{N}=4$ Poincarè superalgebra, which results in the following conditions on the functions involved

$$\imath\left\{f_1,\bar f_1\right\}=f_1\bar f_2+\bar f_1 f_2 \quad\Leftrightarrow\quad \{\sqrt{H},\Phi_1\}=R\cos\Phi_2,$$

with the Hamiltonian $\mathcal{H}$ acquiring the form

$$\mathcal{H}=f_1\bar f_1+\imath\left\{f_1,\bar f_1\right\}\sum_{a=1}^{\mathcal{N}/2}\psi_a\bar\psi^a+\frac{\imath}{2}\left(\left\{f_1,\bar f_2\right\}+\left\{f_2,\bar f_1\right\}\right)\left(\sum_{a=1}^{\mathcal{N}/2}\psi_a\bar\psi^a\right)^2$$

Thus, we get the $\mathcal{N}=4$ supersymmetric mechanics parametrized by two arbitrary functions $\Phi_{1,2}$.

We have shown that the supercharges with cubic fermionic terms allow to construct $\mathcal{N}=4$ supersymmetric mechanics with two functional degrees of freedom, $\mathcal{N}=6$ supersymmetric mechanics with single functional degree of freedom, and $\mathcal{N}=8$ supersymmetric mechanics without any functional freedom. The supercharges with fifth-order fermionic terms will lead to the $\mathcal{N}=6$ supersymmetric mechanics with three functional degrees of freedom and to $\mathcal{N}=8$ supersymmetric mechanics with two functional degrees of freedom. Furthermore, one can expect that the supercharges with seventh-order fermionic terms could lead to the $\mathcal{N}=8$ supersymmetric mechanics with four functional degrees of freedom and so on.

It is easy to deduce that for the construction of $\mathcal{N}=10,12,\ldots 2k$ superextensions of initial Hamiltonian we should choose the following ansatzes for the supercharges

$$Q_a=f_1(z,\bar z)\psi_a+\sum_{l=1}^{\mathcal{N}/2}f_{l+1}(z,\bar z)\psi_a\left(\sum_{b=1}^{\mathcal{N}/2}\psi_b\bar\psi^b\right)^l,$$

$$\bar Q^a=\bar f_1(z,\bar z)\bar\psi^a+\sum_{l=1}^{\mathcal{N}/2}\bar f_{l+1}(z,\bar z)\bar\psi^a\left(\sum_{b=1}^{\mathcal{N}/2}\psi_b\bar\psi^b\right)^l,$$

with $a,b=1,\ldots,\mathcal{N}/2k$. Then, requiring that they form Poincarè superalgebra we will get the family of $\mathcal{N}=2k$ supersymmetric Hamiltonians parameterized by k arbitrary real functions.

By specifying the relevant formulas for Euler top, we obtain a family of $\mathcal{N}=2k$ supersymmetric extensions for Euler top as well

Chapter 3

In this chapter, we propose the description of superintegrable models with dynamical $so(1,2)$ symmetry, as well as the generic superintegrable deformations of oscillator and Coulomb systems, in terms of higher-dimensional Klein model (serving as the non-compact analog of complex projective space), considered as the phase space.

N-dimensional complex projective space $\mathbb{CP}^N$ and its non-compact analog $\widetilde{\mathbb{CP}}^N$. They can be equipped with the $su(N+1)$-invariant (for the compact case) and the $su(1.N)$ invariant (for the non-compact case) Kähler metrics, known as the Fubini-Study metrics. These metrics and respective Kähler potentials are defined by the expressions (with the upper sign corresponding to $\mathbb{CP}^N$, and the lower sign to $\widetilde{\mathbb{CP}}^N$)

$$g_{a\bar{b}}dz^a d\bar{z}^b = \frac{g\,dz d\bar{z}}{1 \pm z\bar{z}} \mp \frac{g(\bar{z}dz)(zd\bar{z})}{(1 \pm z\bar{z})^2}, \quad \mathcal{K} = \pm\, g \log(1 \pm z\bar{z}),$$

as well as the inverse metrics and Poisson brackets given by them

$$g^{\bar{a}b} = (1 \pm z\bar{z})(\bar{z}^a z^b \pm \delta^{\bar{a}b}), \qquad \{z^a, \bar{z}^b\} = \frac{\imath}{g}(1 \pm z\bar{z})(z^a\bar{z}^b \pm \delta^{a\bar{b}})\,.$$

This spaces were obtained by the reduction of (pseudo)Euclidean space $\mathbb{C}^{1,N} \cong U(1,N)$ ($\mathbb{C}^{N+1} \cong U(N+1)$)by the action of $U(1)$ generator. So, g from the above formulae is a constant to which we have set the $U(1)$ generator equal.

We consider N-dimensional analog of the Klein model of non-compact complex projective space $\widetilde{\mathbb{CP}}^N$. The metric and Kähler potential are given by

$$ds^2 = \frac{g[dw + \imath\bar{z}^\alpha dz^\alpha][d\bar{w} - \imath z^\beta d\bar{z}^\beta)]}{[\imath(w-\bar{w}) - z^\gamma\bar{z}^\gamma]^2} + \frac{g\,dz^\alpha d\bar{z}^\alpha}{\imath(w-\bar{w}) - z^\gamma\bar{z}^\gamma},$$

$$\mathcal{K} = -\,g \log\left[\imath(w-\bar{w}) - z^\gamma\bar{z}^\gamma\right], \qquad \alpha,\beta,\gamma = 1,\ldots N-1.$$

The poisson brackets are given by

$$\{w,\bar{w}\} = -A(w-\bar{w}), \quad \{w,\bar{z}^\alpha\} = A\bar{z}^\alpha, \quad \{z^\alpha,\bar{z}^\beta\} = \imath A\delta^{\alpha\bar{\beta}}; \qquad A := \frac{\imath(w-\bar{w}) - z^\gamma\bar{z}^\gamma}{g}\,.$$

The isometry generators are

$$H = \frac{w\bar{w}}{A}, \quad K = \frac{1}{A}, \quad D = \frac{w+\bar{w}}{A}, \quad H_\alpha = \frac{\bar{z}^\alpha w}{A}, \quad K_\alpha = \frac{\bar{z}^\alpha}{A}, \quad H_{\alpha\bar{\beta}} = \frac{\bar{z}^\alpha z^\beta}{A}\,.$$

These generators are forming $su(1.N)$ algebra

$$
\begin{aligned}
&\{H,K\} = -D, \quad \{H,D\} = -2H, \quad \{K,D\} = 2K,\\
&\{H,K_\alpha\} = -H_\alpha, \quad \{H,H_\alpha\} = \{H,H_{\alpha\bar\beta}\} = 0,\\
&\{K,H_\alpha\} = K_\alpha, \quad \{K,K_\alpha\} = \{K,H_{\alpha\bar\beta}\} = 0,\\
&\{D,K_\alpha\} = -K_\alpha, \quad \{D,H_\alpha\} = H_\alpha, \quad \{D,H_{\alpha\bar\beta}\} = 0,\\
&\{K_\alpha,K_\beta\} = \{H_\alpha,H_\beta\} = \{K_\alpha,H_\beta\} = 0,\\
&\{K_\alpha,\bar K_\beta\} = -\imath K\delta_{\alpha\bar\beta}, \quad \{H_\alpha,\bar H_\beta\} = -\imath H\delta_{\alpha\bar\beta},\\
&\{H_{\alpha\bar\beta},H_{\gamma\bar\delta}\} = \imath(H_{\alpha\bar\delta}\delta_{\gamma\bar\beta} - H_{\gamma\bar\beta}\delta_{\alpha\bar\delta}),\\
&\{K_\alpha,H_{\beta\bar\gamma}\} = -\imath K_\beta\delta_{\alpha\bar\gamma}, \quad \{H_\alpha,H_{\beta\bar\gamma}\} = -\imath H_\beta\delta_{\alpha\bar\gamma},\\
&\{K_\alpha,\bar H_\beta\} = H_{\alpha\bar\beta} + \frac{1}{2}\left(g + \sum_\gamma H_{\gamma\bar\gamma} - \imath D\right)\delta_{\alpha\bar\beta}\,.
\end{aligned}
$$

The generators H, K, D define the conformal algebra $su(1.1) = so(1.2)$, and the generators $H_{\alpha\bar\beta}$ define the algebra $u(N-1)$.

It is seen that

- the Hamiltonian H has two sets of constants of motion $H_{\alpha\bar N}$ and $H_{\alpha\bar\beta}$, therefore it defines a superintegrable system;
- the Hamiltonian K has two sets of constants of motion as well, H_α and $H_{\alpha\bar\beta}$. Thus, it defines the superintegrable system as well;
- the triples $(H, H_{\alpha\bar N}, H_{\alpha\bar\beta})$ and $(K, H_\alpha, H_{\alpha\bar\beta})$ transform into each other within discrete transformation

$$
(w, z^\alpha) \to (-\frac{1}{w}, \frac{z^\alpha}{w}) \Rightarrow
$$
$$
D \to -D, (H, H_\alpha, H_{\alpha\bar\beta}) \to (K, -K_\alpha, H_{\alpha\bar\beta}), (K, K_\alpha, H_{\alpha\bar\beta}) \to (H, H_\alpha, H_{\alpha\bar\beta})\,.
$$

Adding to the Hamiltonian H the appropriate function of K, we get the superintegrable oscillator- and Coulomb-like systems.

We define the oscillator-like Hamiltonian by the expression

$$
H_{osc} = H + \omega^2 K
$$

and introduce the following generators

$$
A_\alpha = H_\alpha + \imath\omega K_\alpha \quad B_\alpha = H_\alpha - \imath\omega K_\alpha\,.
$$

Here are some relevant commutation relations:

$$\{A_\alpha, \bar{A}_\beta\} = -\imath\big(H_{osc} - \omega(g + \sum_{\gamma=1}^{N-1} H_{\gamma\bar{\gamma}})\big)\delta_{\alpha\bar{\beta}} + 2\imath\omega H_{\alpha\bar{\beta}},$$

$$\{B_\alpha, \bar{B}_\beta\} = -\imath\big(H_{osc} + \omega(g + \sum_{\gamma=1}^{N-1} H_{\gamma\bar{\gamma}})\big)\delta_{\alpha\bar{\beta}} - 2\imath\omega H_{\alpha\bar{\beta}},$$

$$\{A_\alpha, \bar{B}_\beta\} = -\imath\delta_{\alpha\bar{\beta}}\big(H_{osc} - 2\omega^2 K + \imath\omega D\big),$$

$$\{H_{osc}, A_\alpha\} = -\imath\omega A_\alpha, \quad \{H_{osc}, B_\alpha\} = \imath\omega B_\alpha, \quad \{\bar{H}_{osc}, H_{\alpha\bar{\beta}}\} = 0.$$

Then we immediately deduce that the Hamiltonian besides $H_{\alpha\bar{\beta}}$, has the additional constants of motion which provide the system by the maximal superintegrability property

$$M_{\alpha\beta} = A_\alpha B_\beta = H_\alpha H_\beta + \omega^2 K_\alpha K_\beta + \imath\omega(K_\alpha H_\beta - H_\alpha K_\beta) = \frac{\bar{z}^\alpha \bar{z}^\beta}{A^2}(w^2 + \omega^2)$$

with

$$\{H_{osc}, M_{\alpha\beta}\} = 0.$$

We define the Coulomb-like Hamiltonian with the additional constants of motion which provide the system by the maximal superintegrability property as follows

$$H_{Coul} = H - \frac{\gamma}{\sqrt{2K}}, \quad R_\alpha = H_{\alpha\bar{N}} + \imath\gamma\frac{H_\alpha}{(g + \sum_{\gamma=1}^{N-1} H_{\gamma\bar{\gamma}})\sqrt{2K}} \ ,$$

with

$$\{H_{Coul}, R_\alpha\} = \{H_{Coul}, H_{\alpha\bar{\beta}}\} = 0.$$

One can transit to canonical coordinates

$$w = \frac{p_r}{r} - \imath\frac{\sqrt{2\mathcal{I}}}{r^2}, \quad z^\alpha = \frac{\sqrt{2\pi_\alpha}}{r}e^{\imath\varphi_\alpha}, \qquad \{r, p\} = 1, \quad \{\varphi_\alpha, \pi_\alpha\} = 1,$$

where

$$\mathcal{I} = \frac{1}{2}\left(\sum_{\alpha=1}^{N-1}\pi_\alpha + g\right)^2.$$

In these terms the generators of conformal algebra take the form of conformal mechanics with separated "radial" and "angular" parts

$$H = \frac{p_r^2}{2} + \frac{\mathcal{I}}{r^2}, \quad K = \frac{r^2}{2}, \quad D = p_r r\,.$$

And the rest of generators

$$H_\alpha = \sqrt{2\pi_\alpha}\left(\frac{p_r}{2} - \imath\,\frac{\pi + g}{2r}\right)\mathrm{e}^{-\imath\varphi_\alpha}, \quad K_\alpha = r\sqrt{\frac{\pi_\alpha}{2}}\mathrm{e}^{-\imath\varphi_\alpha}, \quad H_{\alpha\bar\beta} = \sqrt{\pi_\alpha\pi_\beta}\mathrm{e}^{-\imath\left(\varphi_\alpha - \varphi_\beta\right)}.$$

And finally let us write down the oscillator and Coulomb Hamiltonians in these coordinates

$$H_{osc} = \frac{p_r^2}{2} + \frac{\mathcal{I}}{r^2} + \frac{\omega^2 r^2}{2}, \qquad H_{Coul} = \frac{p_r^2}{2} + \frac{\mathcal{I}}{r^2} - \frac{\gamma}{r}.$$

Chapter 4.

In this chapter we consider systems with $su(1,N\,|\,M)$-symmetric $(N\,|\,M)_{\mathbb{C}}$-dimensional Kähler phase space and relate their symmetries with the isometry generators of the super-Kähler structure.

We consider "non-compact projective superspace" $\widetilde{\mathbb{CP}}^{N|M}$, parameterized by N complex bosonic coordinates w, z^α, where $\alpha = 1,...,N-1$, and M complex fermionic coordinates θ^A, where $A = 1,...,M$. They obey the following commutation relations

$$\{w,\bar w\} = -A(w-\bar w), \quad \{z^\alpha,\bar z^\beta\} = \imath A\,\delta^{\alpha\bar\beta}, \quad \{\theta^A,\bar\theta^B\} = A\,\delta^{A\bar B},$$
$$\{w,\bar z^\alpha\} = A\,\bar z^\alpha, \quad \{w,\bar\theta^A\} = A\,\bar\theta^A,$$

where $$A = \frac{1}{g}\left(\imath(w-\bar w) - \sum_{\gamma=1}^{N-1} z^\gamma\bar z^\gamma + \imath\sum_{C=1}^{M}\theta^C\bar\theta^C\right).$$

Here as well, g is a constant, playing analogous role as in previous chapter. The algebra above is defined by the following super-Kähler structure

$$\mathcal{K} = -\,g\,\log[\imath(w-\bar w) - z^\alpha\bar z^\alpha + \imath\theta^A\bar\theta^A].$$

The isometry algebra of this space is $su(N,1\,|\,M)$. It is defined by the following Killing potentials

$$H = \frac{w\bar w}{A}, \quad K = \frac{1}{A}, \quad D = \frac{w+\bar w}{A}, \quad H_\alpha = \frac{\bar z^\alpha w}{A}, \quad K_\alpha = \frac{\bar z^\alpha}{A}, \quad h_{\alpha\bar\beta} = \frac{\bar z^\alpha z^\beta}{A},$$
$$Q_A = \frac{\bar\theta^A w}{A}, \quad S_A = \frac{\bar\theta^A}{A}, \quad \Theta_{A\bar\alpha} = \frac{\bar\theta^A z^\alpha}{A}, \quad R_{A\bar B} = \imath\,\frac{\bar\theta^A\theta^B}{A}.$$

The generators (Killing potentials) form su(1,N|M) superalgebra. For the convenience it is divided into three sectors: "bosonic", "fermionic" and "mixed" ones.

The bosonic sector is direct product of the $su(1,N)$ algebra defined by the generators H, D, K and the $u(M)$ algebra defined by the R-symmetry generators. Explicitly, the $su(1.N)$ algebra is given by the relations

$$
\begin{aligned}
&\{H, K\} = -D, \quad \{H, D\} = -2H, \quad \{K, D\} = 2K,\\
&\{H, K_\alpha\} = -H_\alpha, \quad \{H, H_\alpha\} = \{H, H_{\alpha\bar\beta}\} = 0,\\
&\{K, H_\alpha\} = K_\alpha, \quad \{K, K_\alpha\} = \{K, H_{\alpha\bar\beta}\} = 0,\\
&\{D, K_\alpha\} = -K_\alpha, \quad \{D, H_\alpha\} = H_\alpha, \quad \{D, H_{\alpha\bar\beta}\} = 0,\\
&\{K_\alpha, K_\beta\} = \{H_\alpha, H_\beta\} = \{K_\alpha, H_\beta\} = 0,\\
&\{K_\alpha, \bar K_\beta\} = -\imath K\delta_{\alpha\bar\beta}, \quad \{H_\alpha, \bar H_\beta\} = -\imath H\delta_{\alpha\bar\beta},\\
&\{H_{\alpha\bar\beta}, H_{\gamma\bar\delta}\} = \imath(H_{\alpha\bar\delta}\delta_{\gamma\bar\beta} - H_{\gamma\bar\beta}\delta_{\alpha\bar\delta}),\\
&\{K_\alpha, H_{\beta\bar\gamma}\} = -\imath K_\beta\delta_{\alpha\bar\gamma}, \quad \{H_\alpha, H_{\beta\bar\gamma}\} = -\imath H_\beta\delta_{\alpha\bar\gamma},\\
&\{K_\alpha, \bar H_\beta\} = H_{\alpha\bar\beta} + \frac{1}{2}\left(g + \sum_{\gamma=1}^{N-1} H_{\gamma\bar\gamma} + \sum_{C=1}^{M} R_{C\bar C} - \imath D\right)\delta_{\alpha\bar\beta}\,.
\end{aligned}
$$

The R-symmetry generators form $u(M)$ algebra and commute with all generators of $su(1,N)$:

$$
\{R_{A\bar B}, R_{C\bar D}\} = \imath(R_{A\bar D}\delta_{C\bar B} - R_{C\bar B}\delta_{A\bar D}), \quad \{R_{A\bar B}, (H; K; D; K_\alpha; H_\alpha; h_{\alpha\bar\beta})\} = 0.
$$

The Poisson brackets between fermionic generators are as follows

$$
\begin{aligned}
&\{S_A, \bar S_B\} = K\delta_{A\bar B}, \quad \{Q_A, \bar Q_B\} = H\delta_{A\bar B},\\
&\{S_A, \bar Q_B\} = -\imath R_{A\bar B} + \frac{\imath}{2}\left(g + \sum_{\gamma=1}^{N-1} H_{\gamma\bar\gamma} + \sum_{C=1}^{M} R_{C\bar C} - \imath D\right)\delta_{A\bar B},\\
&\{\Theta_{A\bar\alpha}, \bar\Theta_{B\bar\beta}\} = R_{A\bar B}\delta_{\beta\bar\alpha} + H_{\beta\bar\alpha}\delta_{A\bar B},\\
&\{S_A, \bar\Theta_{B\bar\alpha}\} = K_\alpha\delta_{A\bar B}, \quad \{Q_A, \bar\Theta_{B\bar\alpha}\} = H_\alpha\delta_{A\bar B},\\
&\{S_A, S_B\} = \{Q_A, Q_B\} = \{\Theta_{A\bar\alpha}, \Theta_{B\bar\beta}\} = \{S_A, Q_B\} = \{S_A, \Theta_{B\bar\alpha}\} = \{Q_A, \Theta_{B\bar\alpha}\} = 0.
\end{aligned}
$$

Hence, the functions Q_A play the role of supercharges for the Hamiltonian H, and the functions S_A define the supercharges of the Hamiltonian K playing the role of generator of conformal boosts. The mixed sector is given by the relations

$$
\begin{aligned}
&\{H,Q_A\}=\{H,\Theta_{A\bar\alpha}\}=0,\quad \{H,S_A\}=-Q_A,\\
&\{K,S_A\}=\{K,\Theta_{A\bar\alpha}\}=0,\quad \{K,Q_A\}=S_A,\\
&\{D,S_A\}=-S_A,\quad \{D,Q_A\}=Q_A,\quad \{D,\Theta_{A\bar\alpha}\}=0\\
&\{Q_A,\bar K_\alpha\}=-\Theta_{A\bar\alpha},\quad \{Q_A,(H_\alpha;\bar H_\alpha;\bar K_\alpha;H_{\alpha\bar\beta}\}=0,\\
&\{S_A,\bar H_\alpha\}=\Theta_{A\bar\alpha},\quad \{S_A,(K_\alpha;\bar K_\alpha;H_\alpha;H_{\alpha\bar\beta})\}=0,\\
&\{\Theta_{A\bar\alpha},K_\beta\}=\imath S_A\delta_{\beta\bar\alpha},\quad \{\Theta_{A\bar\alpha},H_\beta\}=\imath Q_A\delta_{\beta\bar\alpha},\quad \{\Theta_{A\bar\alpha},h_{\beta\bar\gamma}\}=\imath\Theta_{A\bar\gamma}\delta_{\beta\bar\alpha},\\
&\{\Theta_{A\bar\alpha},\bar H_\alpha\}=\{\Theta_{A\bar\alpha},\bar K_\alpha\}=0,\\
&\{S_A,R_{B\bar C}\}=-\imath S_B\delta_{A\bar C},\quad \{Q_A,R_{B\bar C}\}=-\imath Q_B\delta_{A\bar C},\quad \{\Theta_{A\bar\alpha},R_{B\bar C}\}=-\imath\Theta_{B\bar\alpha}\delta_{A\bar C}\,.
\end{aligned}
$$

Looking to the all Poisson bracket relations together we conclude that

- The bosonic functions H_α, $h_{\alpha\bar\beta}$, and the fermionic functions Q_A, $\Theta_{A\bar\alpha}$ commute with the Hamiltonian H and thus, provide it by the superintegrability property;

- The bosonic functions K_α, $h_{\alpha\bar\beta}$ and the fermionic functions S_A, $\Theta_{A\bar\alpha}$ commute with the generator K. Hence, the Hamiltonian K defines the superintegrable system as well.

- The triples $(H,H_\alpha,Q_A,)$ and $(K,K_\alpha,S_A,)$ transform into each other under the discrete transformation

$$
(w,z^\alpha,\theta^A)\to(-\frac{1}{w},\frac{z^\alpha}{w},\frac{\theta^A}{w})\quad\Rightarrow
$$
$$
D\to -D,\quad (H,H_\alpha,Q_A,)\to(K,-K_\alpha,-S_A),\quad (K,K_\alpha,S_A)\to(H,H_\alpha,Q_A,)\,.
$$

- The functions $h_{\alpha\bar\beta},\Theta_{A\bar\alpha}$ are invariant under discrete transformation. Moreover, they appear to be constants of motion both for H and K. Hence, they remain to be constants of motion for any Hamiltonian being the functions of H,K. In particular, adding to the Hamiltonian H the appropriate function of K, we get the superintegrable oscillator- and Coulomb-like systems with dynamical superconformal symmetry ;

- The superalgebra $su(1,N\,|\,M)$ admits 5-graded decomposition

$$
su(1,N\,|\,M)=\mathfrak{f}_{-2}\oplus\mathfrak{f}_{-1}\oplus\mathfrak{f}_0\oplus\mathfrak{f}_{+1}\oplus\mathfrak{f}_{+2}
$$

with

$$
\left[\mathfrak{f}_i,\mathfrak{f}_j\right]\subseteq\mathfrak{f}_{i+j}\quad\text{for } i,j\in\{-2,-1,0,1,2\},
$$

where $\mathfrak{f}_i=0$ for $|\,i\,|>2$ is understood. The subset $\mathfrak{f}_0$ includes the generators $D,h_{\alpha\bar\beta},\Theta_{A\bar\alpha},\overline{\Theta}A\,\bar\alpha,R_{A\bar B}$, the subsets $\mathfrak{f}_{-2}$ and $\mathfrak{f}_2$ contain only generators H and K, respectively, while the subsets $\mathfrak{f}_{-1}$ and $\mathfrak{f}_1$ contain the generators $H_\alpha,\bar H_\alpha,Q_A,\bar Q_A$ and $K_\alpha,\bar K_\alpha,S_A,\bar S_A$.

One can transform complex coordinates to canonical ones as follows

$$w = \frac{p_r}{r} - \imath \frac{\sqrt{2\mathcal{I}}}{r^2}, \quad z^\alpha = \frac{\sqrt{2\pi_\alpha}}{r} \mathrm{e}^{\imath\varphi_\alpha}, \quad \theta^A = \frac{\sqrt{2}}{r}\chi^A,$$

with $\sqrt{2\mathcal{I}} = g + \sum_{\alpha=1}^{N-1} \pi_\alpha + \sum_{A=1}^{M} \imath\bar\chi^A \chi^A$ being the Casimir element, and obeying the following Poisson relations

$$\{r, p_r\} = 1, \quad \{\varphi_\alpha, \pi_\beta\} = \delta_{\alpha\beta}, \quad \{\chi^A, \bar\chi^B\} = \delta^{A\bar B}.$$

The isometry generators take the form

$$H = \frac{p_r^2}{2} + \frac{I^2}{2r^2}, \quad K = \frac{r^2}{2}, \quad D = p_r r,$$

$$H_\alpha = \sqrt{\frac{\pi_\alpha}{2}} \mathrm{e}^{-\imath\varphi_\alpha} \left(p_r - \imath \frac{I}{r}\right), \quad K_\alpha = r\sqrt{\frac{\pi_\alpha}{2}} \mathrm{e}^{-\imath\varphi_\alpha}, \quad H_{\alpha\bar\beta} = \sqrt{\pi_\alpha \pi_\beta} \mathrm{e}^{-\imath(\varphi_\alpha - \varphi_\beta)},$$

$$Q_A = \frac{\bar\chi^A}{\sqrt{2}} \left(p_r - \imath \frac{\sqrt{2\mathcal{I}}}{r}\right), \quad S_A = \frac{\bar\chi^A}{\sqrt{2}} r, \quad \Theta_{A\bar\alpha} = \bar\chi^A \sqrt{\pi_\alpha} \mathrm{e}^{\imath\varphi_\alpha}, \quad R_{A\bar B} = \imath\bar\chi^A \chi^B.$$

We define the supersymmetric oscillator-like system with the phase space $\widetilde{\mathbb{CP}}^{N|M}$ by the Hamiltonian

$$H_{osc} = H + \omega^2 K.$$

This system possesses the $u(N)$ symmetry given by the generators $h_{\alpha\bar\beta}$ defined in (among them $N-1$ constants of motion π_α are functionally independent), the $U(M)$ R-symmetry given by the generators $R_{A\bar B}$ as well as $N-1$ hidden symmetries given by the generators

$$M_{\alpha\beta} = (H_\alpha + \imath\omega K_\alpha)(H_\beta - \imath\omega K_\beta) = \frac{\bar z^\alpha \bar z^\beta}{A^2}(w^2 + \omega^2) \ : \quad \{H_{osc}, M_{\alpha\beta}\} = 0,$$

with symmetry algebra

$$\{h_{\alpha\bar\beta}, M_{\gamma\delta}\} = \imath\left(M_{\alpha\delta}\delta_{\gamma\bar\beta} + M_{\gamma\alpha}\delta_{\delta\bar\beta}\right), \quad \{M_{\alpha\beta}, M_{\gamma\delta}\} = 0,$$

$$\{M_{\alpha\beta}, \bar M_{\gamma\delta}\} = \imath\left(4\omega^2 I h_{\alpha\bar\delta} h_{\beta\bar\gamma} - \frac{M_{\alpha\beta}\bar M_{\gamma\delta}}{h_{\alpha\bar\gamma}}\delta_{\alpha\bar\gamma} - \frac{M_{\alpha\beta}\bar M_{\gamma\delta}}{h_{\alpha\bar\delta}}\delta_{\alpha\bar\delta} - \frac{M_{\alpha\beta}\bar M_{\gamma\delta}}{h_{\beta\bar\gamma}}\delta_{\beta\bar\gamma} - \frac{M_{\alpha\beta}\bar M_{\gamma\delta}}{h_{\beta\bar\delta}}\delta_{\beta\bar\delta}\right).$$

Let us choose the following Ansatz for supercharges

$$\mathcal{Q}_A = Q_A + \omega C_{AB}\bar S_B,$$

with constant matrix C_{AB} obeying the condition $C_{AB} + C_{BA} = 0, \quad C_{AB}\overline{C}_{BD} = -\delta_{A\bar{D}}$.
Calculating Poisson brackets we get

$$\{\mathcal{Q}_A, \bar{\mathcal{Q}}_B\} = H_{osc}\delta_{AB}, \qquad \{\mathcal{Q}_A, \mathcal{Q}_B\} = -\imath\omega\mathcal{G}_{AB}, \qquad \{\bar{\mathcal{Q}}_A, \bar{\mathcal{Q}}_B\} = \imath\omega\bar{\mathcal{G}}_{AB},$$

where $\mathcal{G}_{AB} := C_{AC}R_{B\bar{C}} + C_{BC}R_{A\bar{C}}, \qquad \mathcal{G}_{\bar{A}\bar{B}} := \bar{\mathcal{G}}_{AB} = \bar{C}_{AC}R_{C\bar{B}} + \bar{C}_{BC}R_{C\bar{A}}$.

Then we get that the algebra of generators $\mathcal{Q}_A, \mathcal{H}_{osc}, \mathcal{R}_A^B$ is closed indeed:

$$\begin{aligned}
&\{\mathcal{Q}_A, H_{osc}\} = \omega C_{AB}\mathcal{Q}_B, \qquad \{\mathcal{G}_{AB}, H_{osc}\} = 0,\\
&\{\mathcal{Q}_A, \mathcal{G}_{BC}\} = \imath(C_{AB}\mathcal{Q}_C + C_{AC}\mathcal{Q}_B),\\
&\{\mathcal{Q}_A, \bar{\mathcal{G}}_{BC}\} = -\imath(\bar{C}_{BD}\mathcal{Q}_D\delta_{A\bar{C}} + \bar{C}_{CD}\mathcal{Q}_D\delta_{A\bar{B}}).\\
&\{\mathcal{G}_{AB}, \mathcal{G}_{CD}\} = \imath(C_{AD}\mathcal{G}_{BC} + C_{AC}\mathcal{G}_{BD} + C_{BD}\mathcal{G}_{AC} + C_{BC}\mathcal{G}_{AD}),\\
&\{\mathcal{G}_{AB}, \bar{\mathcal{G}}_{CD}\} = \imath(\bar{C}_{DN}\delta_{AC} + \bar{C}_{CN}\delta_{AD})\mathcal{G}_{NB} + \imath(\bar{C}_{DN}\delta_{BC} + \bar{C}_{CN}\delta_{BD})\mathcal{G}_{NA}.
\end{aligned}$$

and surely,

$$\{\mathcal{Q}_A, H_{osc} + \frac{\imath\omega}{2}\sum_B \mathcal{G}_{BB}\} = 0.$$

Hence, for the $M = 2k$ the above oscillator-like system possesses deformed $\mathcal{N} = 4k$ supersymmetry.

Now, let us construct on the phase space $\widetilde{\mathbb{CP}}^{N|M}$ the Coulomb-like system given by the Hamiltonian

$$H_{Coul} = H + \frac{\gamma}{\sqrt{2K}},$$

The bosonic constants of motion of this system are given by the $u(N-1)$ symmetry generators $h_{\alpha\bar{\beta}}$, and by the $N-1$ additional constants of motion

$$R_\alpha = H_\alpha + \imath\gamma\frac{K_\alpha}{I\sqrt{2K}} \quad : \quad \{H_{Coul}, R_\alpha\} = \{H_{Coul}, h_{\alpha\bar{\beta}}\} = 0.$$

These generators form the algebra

$$\{R_\alpha, \bar{R}_{\bar{\beta}}\} = -\imath\delta_{\alpha\bar{\beta}}\left(H_{Coul} - \frac{\imath\gamma^2}{2I^2}\right) + \frac{\imath\gamma^2 h_{\alpha\bar{\beta}}}{2I^3},$$
$$\{h_{\alpha\bar{\beta}}, R_\gamma\} = \imath\delta_{\gamma\bar{\beta}}R_\alpha, \quad \{R_\alpha, R_\beta\} = 0.$$

One can expect, that in analogy with oscillator-like system, our Coulomb-like system would possess (deformed) $\mathcal{N} = 2M$-super-Poincarѐ symmetry
for $M = 2k$ and $\gamma > 1$. However, it is not a case.

Indeed, let us choose the following Ansatz for supercharges

$$\mathcal{Q}_A = Q_A + \sqrt{2\gamma}C_{AB}\frac{\bar{S}_B}{(2K)^{3/4}},$$

with the constant matrix C_{AB} obeying the conditions mentioned in oscillator consideration, $M = 2k$ and $\gamma > 0$. Calculating their Poisson brackets we find

$$\{\mathcal{Q}_A, \bar{\mathcal{Q}}_B\} = H_{Coul}\delta_{A\bar{B}} + \frac{3}{2}\frac{\sqrt{2\gamma}}{(2K)^{7/4}}\left(S_A\bar{C}_{BD}S_D + \bar{S}_B C_{AD}\bar{S}_D\right),$$

$$\{\mathcal{Q}_A, \mathcal{Q}_B\} = -\frac{\imath\sqrt{2\gamma}}{2(2K)^{3/4}}(C_{BD}\mathcal{R}_A^D + C_{AC}\mathcal{R}_B^D), \quad \{\mathcal{Q}_A, \mathcal{R}_B^C\} = -\imath\mathcal{Q}_B\delta_{A\bar{C}}.$$

Further calculating the Poisson brackets of $\mathcal{Q}_A$ with the generators appearing in the r.h.s. of the above expressions we get that the superalgebra is not closed. For example,

$$\{\mathcal{Q}_A, H_{Coul}\} = \frac{3\gamma}{(2K)^{3/2}}S_A + \frac{\sqrt{2\gamma}}{(2K)^{3/4}}C_{AB}\left(\bar{\mathcal{Q}}_B - \frac{3}{4K}\bar{S}_B D\right).$$

Hence, proposed supercharges do not yield closed deformation of $\mathcal{N} = 2M$-super-Poincarè algebra.

Chapter 5.

This is a discussion chapter outlining the main results of the thesis. In the *first chapter* we have discussed some basics of Hamiltonian formalism, the geometry of integrability, specially we have considered the use of the Kähler manifold regarded as a phase space of Hamiltonian systems. Some examples of maximally integrable systems and maximally symmetric Kähler (phase) spaces have been illustrated.

In the *second chapter* we formulated the Euler top as a system with phase space $\mathbb{CP}^1$, i.e. as one-dimensional system. Then we proposed the procedure of $\mathcal{N} = 2k$ *á priori* integrable supersymmetrization of a generic one-dimensional systems which provides the family of $\mathcal{N}$-supersymmetric extensions depending on $\mathcal{N}/2$ arbitrary real functions. Thus, we gave the $\mathcal{N} = 2k$ supersymmetric extensions of the Euler top as well.

In the *third chapter* we have shown that the superintegrable generalizations of conformal mechanics, oscillator and Coulomb systems can be naturally described in terms of the non-compact complex projective space considered as a phase space. This observation yields some interesting directions for further studies.

For example, performing the transformation to the higher-dimensional Poincare model, we expect to present the considered models in the Ruijsenaars-Schneider -like form and in this way to find, some superintegrable cases of the Ruijsenaars-Schneider systems, as well as their supersymmetric/superconformal extensions.

Another one is describing the superintegrable deformations of the free particle on the spheres/hyperboloids, and the spherical/hyperbolic oscillators, in a similar way. For this purpose we expect to consider the "κ-deformation" of the Kähler structure of the Klein model, in the spirit of the so-called "κ-deformation approach".

As well as, we are going to undertake the construction of spin-extensions for the mentioned models, opting for the non-compact analogs of complex Grassmannians as phase spaces.

In *Chapter 4* we suggested to construct the $su(1,N|M)$-superconformal mechanics formulating them on phase superspace given by the non-compact analog of complex projective superspace $\widetilde{\mathbb{CP}}^{N|M}$. The $su(1,N|M)$ symmetry generators were defined there as a Killing potentials of $\widetilde{\mathbb{CP}}^{N|M}$. We parameterized this phase space by the specific coordinates allowing to interpret it as a higher-dimensional super-analog of the Lobachevsky plane parameterized by lower half-plane (Klein model). Then we transited to the canonical coordinates corresponding to the known separation of the "radial" and "angular" parts of (super)conformal mechanics. Relating the "angular" coordinates with action-angle variables we demonstrated that proposed scheme allows to construct the $su(1,N|M)$ superconformal extensions of wide class of superintegrable systems. We also proposed the superintegrable oscillator- and Coulomb- like systems with a $su(1,N|M)$ dynamical superalgebra, and found that oscillator-like systems admit deformed $\mathcal{N}=2M$ Poincarè supersymmetry, in contrast with Coulomb-like ones.

In fact, proposed scheme demonstrated the effectiveness of the supersymmetrization via formulation of the initial systems in terms of Kähler phase space and further generalisation of the latter ones. In order to relate considered systems with the conventional ones (with Euclidean configuration spaces) , we restricted ourselves by the non-compact complex projective superspace. So, we are sure that applying the same approach to the conventional (compact) complex projective spaces we can find many new integrable systems as well and construct their unpredictable extended supersymmetric extensions.

Complete **Bibliography** is presented in the thesis.

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Խաստյան Էրիկ Արթուրի

Կելերյան փուլային տարածությունով սուպերսիմետրիկ մեխանիկաներ

Ամփոփագիր

Ատենախոսությունը նվիրված է դասական մեխանիկական (գեր)ինտեգրվող համակարգերի ուսումնասիրությանը, որոնք ունեն Կելերյան փուլային տարածու-թյուն, և դրանց գերհամաչափ ընդհանրացումների կառուցմանը: Ինչպես ցույց է տրվել աշխատանքում, այստեղ օգտագործվող մեթոդները կարող են չափազանց օգտակար լինել տարատեսակ մոդելների գերհամաչափեցման համար: Նման գեր-համաչափ ընդհանրացումների փուլային գերտարածությունները Կելերյան գերբազ-մաձևություններ են, այսինքն՝ Կելերյան բազմաձևություններ, որոնք հագեցած են նաև Գրասմանյան հակակոմուտացվող կոորդինատներով:

Դասական համակարգ Էյլերի հոլը ձևակերպվել է որպես $\mathbb{CP}^1$ փուլային տարածու-թյուն ունեցող համակարգ, այսինքն՝ որպես միաչափ համակարգ: Այնուհետև առաջարկվել է միաչափ համակարգերի *ա պրիորի* ինտեգրելի $\mathcal{N}=2k$գերհամաչափեցման համընդհանուր ընթացակարգ, որը թույլ է տալիս կառուցել $\mathcal{N}$-գերհամաչափ ընդհանրացումների մի ամբողջ ընտանիք, որոնք կախված են $\mathcal{N}/2$ կամայական իրական ֆունկցիաներից: Այսպիսով, տրվել են նաև Էյլերի հոլի $\mathcal{N}=2k$գերհամաչափ ընդհանրացումները:

Ցույց է տրվել, որ կոնֆորմ մեխանիկայի, օսցիլյատորի և Կուլոնյան համակարգի գերինտեգրելի ընդհանրացումները բնականորեն կարելի է նկարագրել ոչ կոմպակտ կոմպլեքս պրոյեկտիվ տարածության միջոցով, որը դիտարկվում է որպես այդ համակարգերի փուլային տարածություն:

Առաջարկվել է կառուցել $su(1,N|M)$ գերկոնֆորմ մեխանիկա՝ ձևակերպելով այն որպես համակարգ, որի փուլային գերտարածությունը $\mathbb{CP}^{N|M}$ կոմպլեքս պրոյեկտիվ գերտարածության ոչ կոմպակտ անալոգն է ($\widetilde{\mathbb{CP}}^{N|M}$): $su(1,N|M)$ համաչափության գեներատորները սահմանվել են որպես $\widetilde{\mathbb{CP}}^{N|M}$-ի Քիլինգի պոտենցիալներ: Փուլային տարածությունը պարամետրիզացվել է հատուկ կոորդինատներով, ինչը թույլ է տվել այն մեկնաբանել որպես Լոբաչևսկու հարթության բարձր չափանի անալոգ, որը պարամերտիզացված է ստորին կիսահարթությամբ (Քլայնի մոդել): Այնուհետև անցում է կատարվել կանոնական կոորդինատների, որոնք համապատասխանում են (գեր)կոնֆորմ մեխանիկայի «ռադիալ» և «անկյունային» մասերի հայտնի անջատմանը: Կապելով «անկյունային» կոորդինատները անկյուն-գործողություն փոփոխականների հետ՝ ցույց է տրվել, որ առաջարկված սխեման թույլ է տալիս կառուցել գերինտեգրելի համակարգերի լայն դասի $su(1,N|M)$ սուպերկոնֆորմ ընդհանրացումներ: Ինչպես նաև առաջարկվել է $su(1,N|M)$ դինամիկ գերհանրահաշիվով օսցիլյատորի և Կուլոնյանի կերպ համակարգեր, և պարզվել, որ օսցիլատորի կերպ համակարգերի դեպքում հնարավոր է կառուցել դեֆորմացված $\mathcal{N}=2M$ Պուանկարեի սուպերսիմետրիզացիա, Կուլոնյանի կերպ համակարգերի դեպքում ոչ:

Хастян Эрик Артурович

Суперсимметричные механики с Кэлеровым фазовым пространством

Резюме

Диссертация посвящена (супер)интегрируемым системам классической механики с Кэлеровым фазовым пространством и построению их суперсимметричных расширений. Как было показано в работе, используемые методы могут оказаться довольно полезными в суперсимметризации различных моделей. Фазовым пространством таких суперсимметричных расширений является Кэлерово супермногообразие, то есть, Кэлерово многообразие насыщенное антикоммутативными Гроссманновыми переменными.

Классическая система волчок Эйлера была сформулирована как система с фазовым пространством $\mathbb{CP}^1$, то есть как одномерная система. Далее была предложена общая процедура суперсимметризации *априори* интегрируемых одномерных систем, позволяющая построить целое семейство $\mathcal{N}$-суперсимметричных расширений, зависящих от $\mathcal{N}/2$ произвольных вещественных функций. Таким образом, также была построена $\mathcal{N}=2k$ -суперсимметризация волчка Эйлера.

Было показано, что суперинтегрируемые обобщения конформной механики, осциллятора и Кулоновой системы естественным образом можно описать с помощью некомпактного комплексного проективного пространства, играющего роль фазового пространства.

Была построена $su(1,N|M)$ суперконформная механика, сформулированная как система с фазовым пространством $\widetilde{\mathbb{CP}}^{N|M}$, являющимся не компактным аналогом комплексного проективного суперпространства. Генераторы $su(1,N|M)$ симметрии были определены как потенциалы Киллинга $\widetilde{\mathbb{CP}}^{N|M}$. Фазовое пространство было параметризовано специальными координатами, позволяющими его интерпретировать как многомерный аналог плоскости Лобачевского, параметризованной нижней полуплоскостью (модель Клейна). Затем был осуществлен переход к каноническим переменным, соответствующим известному разделению на "радиальную" и "угловую" части в (супер)конформной механике. Связывая “угловые” координаты с переменными действие-угол, было показано, что предложенная схема позволяет построить $su(1,N|M)$ суперконформные расширения широкого класса суперинтегрируемых систем. Также были предложены суперконформные расширения осциллятора и Кулоновой системы с динамической супералгеброй $su(1,N|M)$, и было показано, что в случае осциллятора можно построить деформированную $\mathcal{N}=2M$ суперсимметрию Пуанкаре, в отличие от случая Кулона.